Int. J. Solids Structures Vol. 24, No. 9, pp. 859–880, 1988
Printed in Great Britain.

0020–7683/88 \$3.00+ .00

AN EXACT THEORY OF ELASTIC PLATES

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(*Received* 27 *May* 1987)

Abstract—An extension of recent analyses for the transverse motion of an elastic plate under a transverse load is given which shows that the problem of the loading of a thin plate is a singular perturbation one. Expressions are obtained for all the displacements and stress components in terms of the mid-plane transverse displacement, w, and its derivatives. It is shown that in some cases w is the solution of a standard biharmonic problem. Additionally it is shown how to derive other approximations such as those of Kirchhoff, Mindlin and Reissner avoiding internal inconsistencies and how they are related to the exact theory.

1. INTRODUCTION

This paper extends some recent work of Cheng[1] and Gregory and Wan[2] to determine some exact expressions for the displacements in a plate which is subject to a transverse loading. Additionally some observations are made on the formulation of the theories of thin plates due to Mindlin[3], Reissner[4], Kromm[5] and Hencky[6] and the original work of Kirchhoff[7].

In Cheng's theory[1] an infinite order differential equation for the vertical mid-plane displacement can initially be derived where higher than fourth-order derivatives are multiplied by the square of the plate thickness, h. In the limit as $h \to 0$ the classical fourth-order inhomogeneous biharmonic equation of Kirchhoff is recovered. Thus plate theory is confirmed to be a singular perturbation problem. The other theories lead to sixth-order systems involving the mid-plane displacements and rotations and again exhibit singular behaviour as the plate thickness tends to zero.

Gregory and Wan[2] show that if the boundary data are to produce a decaying solution in the plate interior then the interior solution must satisfy certain boundary conditions. Here it is shown that these conditions, with Cheng's exact displacement solution, can be used to deduce a transformation of the mid-plane displacement variable, so that Cheng's infinite order problem is reduced to precisely Kirchhoff's fourth-order equation with associated boundary conditions.

It is also shown how Mindlin theory is related to Kirchhoff's and how the various other approximate theories are connected to the extended Cheng theory.

2. EQUATIONS OF THREE-DIMENSIONAL LINEAR ELASTICITY

For simplicity it will be assumed that the elastic medium is linear and isotropic and that it fills a region of space given by

$$-a \leqslant x \leqslant a, \quad -b \leqslant y \leqslant b, \quad -h/2 \leqslant z \leqslant h/2$$

where x, y, z are Cartesian coordinates and a and b are both much larger than h. The quantity h will henceforth be referred to as the plate thickness. If the medium is subjected to loads, suppose the resulting displacements are u, v, w, the normal stresses are σ_x, σ_y, σ_z

and the shears are τ_{xy}, τ_{xz}, τ_{yz} in the respective Cartesian directions. The equations for the strains are

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{pmatrix}=\begin{pmatrix}u_{,x}\\ v_{,y}\\ w_{,z}\end{pmatrix},\quad \begin{pmatrix}\gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{pmatrix}=\begin{pmatrix}v_{,x}+u_{,y}\\ u_{,z}+w_{,x}\\ w_{,y}+v_{,z}\end{pmatrix}. \tag{1}$$

For a linear isotropic medium the stresses and strains are related by

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{pmatrix}=\frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix},\quad \begin{pmatrix}\gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{pmatrix}=\frac{1}{G}\begin{pmatrix}\tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{pmatrix} \tag{2}$$

where E is Young's modulus, ν Poisson's ratio and $G = E/2(1+\nu)$ is the shear modulus.

The equilibrium equations, in the presence of a body force $\mathbf{F}$, are

$$\sigma_{x,x}+\tau_{xy,y}+\tau_{xz,z}+F_x = \tau_{xy,x}+\sigma_{y,y}+\tau_{yz,z}+F_y = \tau_{xz,x}+\tau_{yz,y}+\sigma_{z,z}+F_z = 0. \tag{3}$$

In these equations ",α" denotes differentiation with respect to α. Equations (2) may be inverted to express the stresses in terms of the strains to give

$$\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}=\frac{E}{(1-2\nu)(1+\nu)}\begin{pmatrix}1-\nu & \nu & \nu\\ \nu & 1-\nu & \nu\\ \nu & \nu & 1-\nu\end{pmatrix}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{pmatrix}$$

$$\begin{pmatrix}\tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{pmatrix}=\frac{E}{2(1+\nu)}\begin{pmatrix}\gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{pmatrix}. \tag{2'}$$

Elimination of the stresses and strains in terms of the displacements from eqns (2′) and (3) yields the simultaneous equations

$$(1-2\nu)\nabla_0^2\mathbf{u}+\text{grad } e_0+\frac{(1-2\nu)}{G}\mathbf{F} = 0 \tag{4}$$

where ∇_0^2 is the three-dimensional Laplacian operator, $\mathbf{u} = (u, v, w)$ and $e_0 = \text{div}\,\mathbf{u}$ (cf. Szilard[8]).

The first step in Cheng's analysis is to express the displacement vector $\mathbf{u}$ in terms of a vector potential $\mathbf{A}$ via the expression

$$\mathbf{u} = \nabla_0^2\mathbf{A}-\frac{1}{2(1-\nu)}\text{ grad div }\mathbf{A}. \tag{5}$$

Cheng does not establish that this is a unique expression for $\mathbf{u}$. From eqn (4) the equation for $\mathbf{u}$ is

$$\frac{-\mathbf{F}}{G} = \frac{2(1-\nu)}{1-2\nu}\text{ grad div }\mathbf{u}-\text{curl}^2\,\mathbf{u},\quad \nu \neq \tfrac{1}{2}$$

as

$$\nabla_0^2\mathbf{u} = \text{grad div }\mathbf{u} - \text{curl}^2\ \mathbf{u}$$

and so $\mathbf{A}$ satisfies

$$\frac{-\mathbf{F}}{G} = \frac{2(1-\nu)}{1-2\nu}\text{ grad div }\nabla_0^2\mathbf{A} - \frac{1}{1-2\nu}\text{ grad div grad div }\mathbf{A} - \text{curl}^2\ \nabla_0^2\mathbf{A}$$

$$= \frac{1}{1-2\nu}\text{ grad div }(\nabla_0^2\mathbf{A} - \text{grad div }\mathbf{A}) + (\text{grad div} - \text{curl}^2)\nabla_0^2\mathbf{A}$$

as curl grad $\phi = 0$ for any smooth scalar function ϕ.

Hence

$$-\frac{\mathbf{F}}{G} = \nabla_0^4\mathbf{A} \tag{6}$$

as $\nabla_0^2\mathbf{A} - \text{grad div }\mathbf{A} = \text{curl}^2\ \mathbf{A}$ and div curl $\mathbf{F} = 0$ for any smooth vector function $\mathbf{F}$. It is not difficult to show that eqns (5) and (6) apply in general, so that an arbitrary solution of the three-dimensional elasticity equations can be expressed in terms of a vector potential.

Equation (5) is a vector form of the general solutions of the three-dimensional equations quoted by Donnell (Table 3.1, pp. 12–14 of Ref. [9]) which were originally published by the Italian mathematician Somigliana in 1894[10].

Note that when $\nu = \frac{1}{2}$, eqn (5) gives

$$\mathbf{u} = \nabla_0^2\mathbf{A} - \text{grad div }\mathbf{A} = -\text{curl}^2\ \mathbf{A}$$

so div $\mathbf{u} = 0$, i.e. $\nu = \frac{1}{2}$ corresponds to the incompressibility condition.

3. PLATE ANALYSIS

In the analysis of plates, series expansions can be developed in terms of the plate thickness. In pure bending problems u and v are odd functions of z and w is an even function of z. In the absence of body forces eqn (6) can be written, as shown by Cheng, as

$$\left(\nabla^4 + 2\nabla^2\frac{\partial^2}{\partial z^2} + \frac{\partial^4}{\partial z^4}\right)\mathbf{A} = 0$$

where ∇ is the two-dimensional Laplacian in x and y. This has formal solutions in z, matching the symmetry condition, in which

$$A_\alpha = \frac{1}{\nabla}\sin(z\nabla)F_\alpha + z\cos(z\nabla)f_\alpha, \quad \alpha = x, y$$

$$A_z = \cos(z\nabla)F_z + \frac{z}{\nabla}\sin(z\nabla)f_z. \tag{7}$$

These contain six functions F_α, f_α of x and y which are needed to fit the remaining boundary conditions. As $\mathbf{u}$ has only three components, three conditions may be imposed arbitrarily.

From eqns (7)

$$\text{div }\mathbf{A} = \frac{1}{\nabla}\sin(z\nabla)(F_{x,x} + F_{y,y} - \nabla^2 F_z + f_z) + z\cos(z\nabla)(f_{x,x} + f_{y,y} + f_z) \tag{8}$$

and so eqn (5) gives

$$w = -\frac{1}{2(1-\nu)}\{\cos(z\nabla)(F_{x,x}+F_{y,y}-\nabla^2F_z+2f_z+f_{x,x}+f_{y,y}) - z\nabla\sin(z\nabla)(f_{x,x}+f_{y,y}+f_z)\}+2\cos(z\nabla)f_z.$$

This expression is considerably simplified if the F's and f's are chosen to satisfy the three additional conditions

$$F_x = -f_x, \quad F_y = -f_y \quad \text{and} \quad \nabla^2F_z = 2f_z.$$

The equation for w then becomes

$$w = 2\cos(z\nabla)f_z - \frac{z\nabla\sin(z\nabla)}{4(1-\nu)\nabla^2}e \tag{9}$$

where $e = -2\nabla^2(f_{x,x}+f_{y,y}+f_z)$.

Also from eqns (5) and (8)

$$u = -2\nabla\sin(z\nabla)f_x + \frac{1}{4(1-\nu)\nabla^2}\left(z\cos(z\nabla)-\frac{\sin(z\nabla)}{\nabla}\right)e_{,x}$$

and

$$v = -2\nabla\sin(z\nabla)f_y + \frac{1}{4(1-\nu)\nabla^2}\left(z\cos(z\nabla)-\frac{\sin(z\nabla)}{\nabla}\right)e_{,y}. \tag{10}$$

The f's can now be related to the displacements on $z = 0$. From eqn (9)

$$w(x,y,0) = 2f_z(x,y) = \bar{w}(x,y)$$

and from eqns (10) as z approaches zero

$$u \to -2z\nabla^2f_x = zu'(x,y), \text{ say}$$

and

$$v \to -2z\nabla^2f_y = zv'(x,y), \text{ say}.$$

With these expressions

$$e = (u'_{,x}+v'_{,y}-\nabla^2\bar{w}) \tag{11}$$

and so the final expressions for the velocity fields are

$$\begin{pmatrix}u\\v\end{pmatrix} = \frac{\sin(z\nabla)}{\nabla}\begin{pmatrix}u'\\v'\end{pmatrix} + \frac{1}{4(1-\nu)\nabla^2}\left(z\cos(z\nabla)-\frac{\sin(z\nabla)}{\nabla}\right)\begin{pmatrix}e_{,x}\\e_{,y}\end{pmatrix}$$

and

$$w = \cos(z\nabla)\bar{w} - \frac{z\nabla\sin(z\nabla)}{4(1-\nu)\nabla^2}e \tag{12}$$

agreeing with Cheng's formulae.

4. EQUATIONS FOR MID-PLANE DISPLACEMENTS FOR A LOADED PLATE

When the plate is loaded the normal stress on the surface may be supposed to have the form

$$\sigma_z(x, y \pm h/2) = \pm\tfrac{1}{2}p(x, y) \tag{13}$$

with the surface shears zero, i.e.

$$\tau_{xz}(x, y, \pm h/2) = \tau_{yz}(x, y, \pm h/2) = 0. \tag{14}$$

It is useful in plate theory to have equations for the bending and twisting moments and shears. In the case of a normally loaded plate satisfying conditions (13), integration of the equilibrium equations through the plate thickness in the absence of a body force yields

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-q_x = \frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-q_y = \frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+p = 0 \tag{15}$$

where

$$M_x = \int_{-h/2}^{h/2} \sigma_x z \,\mathrm{d}z, \quad M_y = \int_{-h/2}^{h/2} \sigma_y z \,\mathrm{d}z$$

are the bending moments

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \,\mathrm{d}z$$

is the twisting moment and

$$q_x = \int_{-h/2}^{h/2} \tau_{xz} \,\mathrm{d}z \quad \text{and} \quad q_y = \int_{-h/2}^{h/2} \tau_{yz} \,\mathrm{d}z$$

are the shears.

Expressions (11) and (12) can be used to determine the moments and shears as

$$M_x = \frac{Eh}{(1+\nu)}\left\{\left(-c+\frac{2s}{h\nabla}\right)\left[\frac{1}{\nabla^2}u'_{,x}+\frac{1}{4(1-\nu)\nabla^2}\left(2\nu e-\frac{1}{\nabla^2}e_{,xx}\right)\right]\right.$$
$$\left.+\left(c-\frac{2s}{h\nabla}+\frac{sh\nabla}{4}\right)\frac{1}{2(1-\nu)\nabla^4}e_{,xx}\right\}$$

$$M_y = \frac{Eh}{1+\nu}\left\{\left(-c+\frac{2s}{h\nabla}\right)\left[\frac{1}{\nabla^2}v'_{,y}+\frac{1}{4(1-\nu)\nabla^2}\left(2\nu e-\frac{1}{\nabla^2}e_{,yy}\right)\right]\right.$$
$$\left.+\left(c-\frac{2s}{h\nabla}+\frac{sh\nabla}{4}\right)\frac{1}{2(1-\nu)\nabla^4}e_{,yy}\right\}$$

$$M_{xy} = \frac{Eh}{1+\nu}\left\{\left(-c+\frac{2s}{h\nabla}\right)\left(\frac{1}{2\nabla^2}(u'_{,y}+v'_{,x})-\frac{1}{4(1-\nu)\nabla^4}e_{,xy}\right)\right.$$
$$\left.+\left(c-\frac{2s}{h\nabla}+\frac{sh\nabla}{4}\right)\frac{1}{2(1-\nu)\nabla^4}e_{,xy}\right\}$$

$$q_x = \frac{Eh}{2(1+\nu)}\left\{\frac{2s}{h\nabla}(\bar{w}_{,x}+u')+\left(c-\frac{2s}{h\nabla}\right)\frac{1}{2(1-\nu)\nabla^2}e_{,x}\right\}$$

and

$$q_y = \frac{Eh}{2(1+\nu)}\left\{\frac{2s}{h\nabla}(\bar{w}_{,y}+v')+\left(c-\frac{2s}{h\nabla}\right)\frac{1}{2(1-\nu)\nabla^2}e_{,y}\right\} \tag{16}$$

where $c = \cos(h\nabla/2)$ and $s = \sin(h\nabla/2)$.

For small h, $2s/h\nabla \to 1$ and $c-2s/h\nabla \to O(h^2)$ so that the moments are of $O(h^3)$ and the shears of $O(h)$. The quantities u' and v' are the rotations. The extra terms in the shears are novel $O(h^2)$ correction terms involving the in-plane strain since

$$e_{,x} = -\nabla^2(\bar{w}_{,x}+u')+(u_{,y}+v_{,x})_{,y}+2u'_{,xx}.$$

An alternative form for the x-shear is thus

$$q_x = \frac{Eh}{2(1+\nu)}\left\{\left(\frac{2s}{h\nabla}-\frac{1}{2(1-\nu)}\left(c-\frac{2s}{h\nabla}\right)\right)(\bar{w}_{,x}+u')+\left(c-\frac{2s}{h\nabla}\right)\frac{1}{2(1-\nu)\nabla^2}(2\varepsilon_{x,x}+\gamma_{xy,y})\right\}.$$

Mindlin's theory, as will be seen later, is a model in which

$$2\varepsilon_{x,x}+\gamma_{xy,y} = \alpha_M\nabla^2(\bar{w}_{,x}+u'), \quad 2\varepsilon_{y,y}+\gamma_{xy,x} = \alpha_M\nabla^2(\bar{w}_{,y}+v')$$

where

$$\lim_{h\to 0}\left(\frac{2s}{h\nabla}-\frac{1}{2(1-\nu)}\left(c-\frac{2s}{h\nabla}\right)(1-\alpha_M)\right) = \frac{\pi^2}{12}$$

i.e.

$$\alpha_M = \frac{2(1-\nu)(12-\pi^2)}{h^2\nabla^2}+O(1).$$

The corresponding factor in Reissner's theory, again see later, is

$$\alpha_R = \frac{2(1-\nu)(12-10)}{h^2\nabla^2}+O(1).$$

Conditions (13) and (14) may be written in terms of the mid-plane displacements. Thus

$$\left(c+\frac{sh\nabla}{4(1-\nu)}\right)(\bar{w}_{,x}+u')-\frac{sh\nabla}{4(1-\nu)\nabla^2}((u'_{,x}+v'_{,y})_{,x}+\nabla^2u') = 0$$

$$\left(c+\frac{sh\nabla}{4(1-\nu)}\right)(\bar{w}_{,y}+v')-\frac{sh\nabla}{4(1-\nu)\nabla^2}(u'_{,x}+v'_{,y})_{,y}+\nabla^2v') = 0$$

and

$$\frac{2s}{h\nabla}((\bar{w}_{,x}+u')_{,x}+(\bar{w}_{,y}+v')_{,y})+\left(c-\frac{2s}{h\nabla}\right)\frac{1}{2(1-\nu)}(u'_{,x}+v'_{,y}-\nabla^2\bar{w})+\frac{2(1+\nu)p}{Eh} = 0. \tag{17}$$

A single equation for $\bar{w}$ may be obtained by differentiating the first equation by x the second by y and by eliminating the combination $u'_{,x}+v'_{,y}$ using the third equation. Alternatively if three differential operators D_1, D_2, D_3 defined by

$$D_1 = 4(1-\nu)c, \quad D_2 = h/\nabla s, \quad D_3 = h/2c + \frac{1-2\nu}{\nabla}s \tag{18}$$

are introduced, the following equations result:

$$\begin{pmatrix} D_1 - D_2\dfrac{\partial^2}{\partial x^2} & -D_2\dfrac{\partial^2}{\partial x\,\partial y} & (D_1+D_2\nabla^2)\dfrac{\partial}{\partial x} \\ -D_2\dfrac{\partial^2}{\partial x\,\partial y} & D_1 - D_2\dfrac{\partial^2}{\partial y^2} & (D_1+D_2\nabla^2)\dfrac{\partial}{\partial y} \\ D_3\dfrac{\partial}{\partial x} & D_3\dfrac{\partial}{\partial y} & \dfrac{4(1-\nu)}{h}D_2\nabla^2 - D_3\nabla^2 \end{pmatrix}\begin{pmatrix} u' \\ v' \\ \bar{w} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{-2p(1-\nu^2)}{E} \end{pmatrix}. \tag{19}$$

This system has the determinant

$$\bar{D} = 4(1-\nu)hD_1\nabla^2\left(\frac{\sin h\nabla}{h\nabla} - 1\right) \tag{20}$$

and so $(u', v', \bar{w})$ satisfy

$$\bar{D}u' = D_1(D_1+D_2\nabla^2)\frac{h^3}{6D}p_{,x}$$

$$\bar{D}v' = D_1(D_1+D_2\nabla^2)\frac{h^3}{6D}p_{,y}$$

and

$$\bar{D}\bar{w} = -D_1(D_1-D_2\nabla^2)\frac{h^3}{6D}p$$

or

$$\frac{6}{h^2\nabla^2}\left(1-\frac{2sc}{h\nabla}\right)\nabla^4 u' = -\left(c+\frac{h\nabla}{4(1-\nu)}s\right)\frac{p_{,x}}{D}$$

$$\frac{6}{h^2\nabla^2}\left(1-\frac{2sc}{h\nabla}\right)\nabla^4 v' = -\left(c+\frac{h\nabla}{4(1-\nu)}s\right)\frac{p_{,y}}{D}$$

and

$$\frac{6}{h^2\nabla^2}\left(1-\frac{2sc}{h\nabla}\right)\nabla^4 \bar{w} = \left(c-\frac{h\nabla}{4(1-\nu)}s\right)\frac{p}{D} \tag{21}$$

where

$$D = Eh^3/12(1-\nu^2).$$

Equation $(21)_3$ is an infinite order differential equation for the normal displacement at the mid-surface, involving the two parameters h^2 and D. In the limit as $h \to 0$ with D finite, eqns (21) reduce to the standard biharmonic equation deduced originally by Kirchhoff

$$\nabla^4\bar{w} = \frac{p}{D}. \tag{22}$$

Thus, depending on the boundary conditions, eqns (21) will exhibit singular perturbation type of behaviour. We will next examine the boundary conditions that must be associated with eqns (21).

5. BOUNDARY CONDITIONS

It is a bit alarming that eqns (21) have infinite order as this suggests that $\bar{w}$ can only be determined if it satisfies an infinite number of boundary conditions. However, for certain types of boundary data it turns out that a transformation of $\bar{w}$ satisfies a transformed problem which is of only fourth order. The key to finding this transformation follows from some recent work of Gregory and Wan[2, 11]. In these papers they show, by use of the Betti–Rayleigh reciprocal theorem, how to obtain a correct set of boundary conditions for classical and higher order plate theories for any admissible set of edge data. Two cases which they analyse in detail concern the semi-infinite plate $x \geqslant 0$, $|y| < \infty$, $|z| \leqslant h/2$ where in their terminology

Case B:

$$\sigma_x(0, y, z) = \tilde{\sigma}_x(y, z), \quad v(0, y, z) = \tilde{v}(y, z), \quad w(0, y, z) = \tilde{w}(y, z) \tag{23}$$

or

Case C:

$$u(0, y, z) = \tilde{u}(y, z), \quad \tau_{xy}(0, y, z) = \tilde{\tau}_y(y, z), \quad \tau_{xz}(0, y, z) = \tilde{\tau}_z(y, z) \tag{24}$$

where $(\tilde{\ })$ denotes a known imposed value. In case B they show that to avoid the boundary data causing a growing disturbance away from the boundary edge it is necessary to have

$$\int_{-h/2}^{h/2} (z\tilde{\sigma}_x + 2Gz\tilde{v}_{,y})\, \mathrm{d}z = 0 \tag{25}$$

and

$$\int_{-h/2}^{h/2} \left[\left(\frac{h^2}{4} - z^2\right)\tilde{w} + \frac{2-v}{3} z^3 \tilde{v}_{,y} + \frac{2-v}{6G} z^3 \tilde{\sigma}_x\right] \mathrm{d}z = 0. \tag{26}$$

The boundary conditions that are appropriate to the interior solution, away from the plate edge, require that the difference between the values of σ_x, v and w on $x = 0$ and the data should satisfy eqns (25) and (26). If expressions (12) are used together with the governing equations, eqns (17), then it can be shown that eqn (25) gives

$$\left(c - \frac{sh\nabla}{4(1-v)}\right)^{-1}\left(1 - \frac{2s}{h\nabla}\right)\frac{6}{h^2\nabla^2}\nabla^2\bar{w} = -\frac{1}{D}\int_{-h/2}^{h/2} (z\tilde{\sigma}_x + 2Gz\tilde{v}_{,y})\, \mathrm{d}z \tag{27}$$

whereas eqn (26) gives

$$\left(c - \frac{sh\nabla}{4(1-v)}\right)^{-1}\left(1 - \frac{2sc}{h\nabla}\right)\frac{6}{h^2\nabla^2}\left(1 - \frac{(2-v)h^2\nabla^2}{8(1-v)}\right)\bar{w}$$
$$= \frac{6}{h^3}\int_{-h/2}^{h/2} \left[\left(\frac{h^2}{4} - z^2\right)\tilde{w} + \frac{(2-v)}{3} z^3 \tilde{v}_{,y} + \frac{2-v}{6G} z^3 \tilde{\sigma}_x\right] \mathrm{d}z. \tag{28}$$

A comparison of eqns (27), (28) and (21) shows that $\bar{w}$ is not the best choice of expansion variable. The variables

$$w^* = \left(c - \frac{sh\nabla}{4(1-v)}\right)^{-1}\left(1 - \frac{2sc}{h\nabla}\right)\frac{6}{h^2\nabla^2}\bar{w} \tag{29}$$

and

$$\begin{pmatrix} u^{*\prime} \\ v^{*\prime} \end{pmatrix} = \left(c - \frac{sh\nabla}{4(1-\nu)}\right)^{-1}\left(1 - \frac{2sc}{h\nabla}\right)\frac{6}{h^2\nabla^2}\begin{pmatrix} u' \\ v' \end{pmatrix} \tag{30}$$

are to be preferred since then eqn $(21)_3$ becomes

$$\nabla^4 w^* = \frac{p}{D} \tag{31}$$

which eqns (27) and (28) imply is to be solved with

$$\nabla^2 w^* \quad \text{and} \quad \left(1 - \frac{(2-\nu)h^2\nabla^2}{8(1-\nu)}\right) w^* \tag{32}$$

prescribed. Equations $(17)_1$ and $(17)_2$ then imply that

$$(u^{*\prime}_{,x} + v^{*\prime}_{,y}) = -\left(c - \frac{sh\nabla}{4(1-\nu)}\right)^{-1}\left(c + \frac{sh\nabla}{4(1-\nu)}\right)\nabla^2 w^*$$

and so

$$\begin{pmatrix} cu^{*\prime} \\ cv^{*\prime} \end{pmatrix} = -\left(c + \frac{sh\nabla}{4(1-\nu)}\right)\begin{pmatrix} w^*_{,x} \\ w^*_{,y} \end{pmatrix} + \frac{sh\nabla}{4(1-\nu)\nabla^2}\begin{pmatrix} (u^{*\prime}_{,x} + v^{*\prime}_{,y})_{,x} \\ (u^{*\prime}_{,x} + v^{*\prime}_{,y})_{,y} \end{pmatrix}$$

i.e.

$$\begin{pmatrix} u^{*\prime} \\ v^{*\prime} \end{pmatrix} = -\left(c - \frac{sh\nabla}{4(1-\nu)}\right)^{-1}\left(c + \frac{sh\nabla}{4(1-\nu)}\right)\begin{pmatrix} w^*_{,x} \\ w^*_{,y} \end{pmatrix}. \tag{33}$$

The structure is now clear, eqn (31) and boundary conditions (32) form part of a *fourth-order* boundary value problem for w^* and eqns (33) are infinite series for the rotations $u^{*\prime}$ and $v^{*\prime}$ which may be computed by differentiation only. To leading order the rotations are the negatives of the appropriate mid-plane gradients. To summarize the displacements, satisfying the exact three-dimensional equations, can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(c - \frac{sh\nabla}{4(1-\nu)}\right)\left(1 - \frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\frac{\sin(z\nabla)}{\nabla}\begin{pmatrix} u^{*\prime} \\ v^{*\prime} \end{pmatrix} + \frac{1}{4(1-\nu)\nabla^2}\left(z\cos(z\nabla) - \frac{\sin(z\nabla)}{\nabla}\right)\begin{pmatrix} e^*_{,x} \\ e^*_{,y} \end{pmatrix}\right\}$$

$$w = \left(c - \frac{sh\nabla}{4(1-\nu)}\right)\left(1 - \frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\cos(z\nabla)w^* - \frac{z\nabla\sin(z\nabla)}{4(1-\nu)\nabla^2}e^*\right\} \tag{34}$$

where $u^{*\prime}$, $v^{*\prime}$ are given by eqns (33), w^* satisfies eqn (31) with appropriate boundary conditions, $c = \cos(h\nabla/2)$, $s = \sin(h\nabla/2)$ and

$$e^* = u^{*\prime}_{,x} + v^{*\prime}_{,y} - \nabla^2 w^*.$$

It is also possible to express the displacements solely in terms of the quantity w^*. The relevant expressions are

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\frac{\sin(z\nabla)}{\nabla}\left(c+\frac{sh\nabla}{4(1-\nu)}\right)\right.$$
$$\left.+\left(z\cos(z\nabla)-\frac{\sin(z\nabla)}{\nabla}\right)\frac{c}{2(1-\nu)}\right\}\begin{pmatrix} w^*_{,x} \\ w^*_{,y} \end{pmatrix}$$

and

$$w = \left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\cos(z\nabla)\left(c-\frac{sh\nabla}{4(1-\nu)}\right)+\frac{z\nabla\sin(z\nabla)c}{2(1-\nu)}\right\}w^*. \tag{35}$$

The expressions for the moments and shears are also much simpler than expressions (16). They are

$$M_x = -D(w^*_{,xx}+\nu\lambda w^*_{,yy})$$
$$M_y = -D(w^*_{,yy}+\nu\lambda w^*_{,xx})$$
$$M_{xy} = -D(1-\nu\lambda)w^*_{,xy}$$
$$q_x = -D\nabla^2 w^*_{,x}$$

and

$$q_y = -D\nabla^2 w^*_{,y} \tag{36}$$

where

$$\lambda = -2\left(1-\frac{2sc}{h\nabla}\right)^{-1}\left(c-\frac{2s}{h\nabla}\right)c \simeq 1-\frac{h^2\nabla^2}{10}+O(h^4\nabla^4).$$

These agree with the formulae of Timoshenko and Woinowsky-Kreiger (p. 102 of Ref. [12]) which are written in terms of $\bar{w}$ and apply for the case $\nabla^4\bar{w} = 0$.

Expressions can also be derived for the stresses. These are

$$\sigma_x = -\frac{E}{1-\nu^2}\left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\frac{\sin(z\nabla)}{\nabla}\left[\left(\frac{1-2\nu}{2}c+\frac{sh\nabla}{4}\right)w^*_{,xx}+\nu c\nabla^2 w^*\right]\right.$$
$$\left.+\frac{c}{2}z\cos(z\nabla)w^*_{,xx}\right\}$$

$$\sigma_y = -\frac{E}{1-\nu^2}\left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\frac{\sin(z\nabla)}{\nabla}\left[\left(\frac{(1-2\nu)}{2}c+\frac{sh\nabla}{4}\right)w^*_{,yy}+\nu c\nabla^2 w^*\right]\right.$$
$$\left.+\frac{c}{2}z\cos(z\nabla)w^*_{,yy}\right\}$$

$$\sigma_z = \frac{E}{2(1-\nu^2)}\left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{z\cos(z\nabla)c-\frac{\sin(z\nabla)}{\nabla}\left(c-\frac{sh\nabla}{2}\right)\right\}\nabla^2 w^*$$

$$\tau_{xy} = \frac{E}{1-\nu^2}\left(1-\frac{2sc}{h\nu}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\sin\frac{(z\nabla)}{\nabla}\left(\frac{(1-2\nu)}{2}c+\frac{sh\nabla}{4}\right)+\frac{z\cos(z\nabla)}{2}c\right\}w^*_{,xy}$$

$$\tau_{xz} = -\frac{E}{1-\nu^2}\left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\frac{sh\nabla}{4}\cos(z\nabla)-\frac{z\nabla}{2}\sin(z\nabla)c\right\}w^*_{,x}$$

$$\tau_{yz} = -\frac{E}{1-\nu^2}\left(1-\frac{2sc}{h\nabla}\right)^{-1}\frac{h^2\nabla^2}{6}\left\{\frac{sh\nabla}{4}\cos(z\nabla)-\frac{z\nabla}{2}\sin(z\nabla)c\right\}w^*_{,y}.$$

These are generalizations of the results of Timoshenko and Woinowsky-Kreiger (p. 103 of Ref. [12]) where σ_z can vary through the plate thickness due to an imposed load. The normal load can be confirmed to be $p/2$ on $z = h/2$ by substitution in the expression for σ_z and the use of the result that $D\nabla^4 w^* = p$. It may also be checked that the stresses satisfy the equilibrium equations exactly.

In case C Gregory and Wan[2] quote the conditions

$$\int_{-h/2}^{h/2} (\tilde{\tau}_{xz} + z\tilde{\tau}_{xy,y} - 2Gz\tilde{u}_{,yy})\ \mathrm{d}z = 0 \tag{37}$$

and

$$\int_{-h/2}^{h/2} \left[4Gz\tilde{u} - z\left(\frac{h^2}{4} - \frac{2-\nu}{6}z^2\right)(4G\tilde{u}_{,yy} - 2\tilde{\tau}_{xy,y}) + \nu z^2 \tau_{xz} \right] \mathrm{d}z = 0. \tag{38}$$

Once more it is possible to deduce conditions on the mid-plane vertical displacement $\bar{w}$. The conditions become

$$\left(c - \frac{sh\nabla}{4(1-\nu)}\right)^{-1}\left(1 - \frac{2sc}{h\nabla}\right)\frac{6}{h^2\nabla^2}\nabla^2\bar{w}_{,x} = -\frac{1}{D}\int_{-h/2}^{h/2} (\tilde{\tau}_{xz} + z\tilde{\tau}_{xy,y} - 2Gz\tilde{u}_{,yy})\ \mathrm{d}z \tag{39}$$

and

$$\left(c - \frac{sh\nabla}{4(1-\nu)}\right)^{-1}\left(1 - \frac{2sc}{h\nabla}\right)\frac{6}{h^2\nabla^2}\left(1 + \frac{\nu}{8(1-\nu)}h^2\nabla^2\right)\bar{w}_{,x}$$
$$= -\frac{12}{h^3}\int_{-h/2}^{h/2} \left[z\tilde{u} - z\left(\frac{h^2}{4} - \frac{2-\nu}{6}z^2\right)\tilde{u}_{,yy} - \frac{\tilde{\tau}_{xy,y}}{2G} + \frac{\nu}{4G}z^2\tilde{\tau}_{xz} \right] \mathrm{d}z. \tag{40}$$

Transformations (29) and (30) give the governing equation as eqn (31), this time with

$$\nabla^2 w^*_{,x} \quad \text{and} \quad \left(1 + \frac{\nu h^2\nabla^2}{8(1-\nu)}\right)w^*_{,x} \tag{41}$$

prescribed. As Gregory and Wan[2] point out, if only the shears are given on the boundary of a thin plate then eqns (39) and (40) can be combined (as $h \to 0$) to give

$$\nabla^2\bar{w}_{,x} + \text{known} = \frac{2G}{D}\left(\int_{-h/2}^{h/2} z\tilde{u}\ \mathrm{d}z\right)_{,yy} = \frac{2G}{D}(-)\frac{h^3}{12}(\bar{w}_{,xyy} - \text{known})$$

i.e.

$$\nabla^2\bar{w}_{,x} + (1-\nu)\bar{w}_{,xyy} = \text{known} \tag{42}$$

which is of course the well-known Kirchhoff condition. It may also be noted that the left-hand sides of boundary conditions (27), (28), (39) and (40) expanded in series of $h^2\nabla^2$ reduce to the results of Gregory and Wan. Since in their case $\nabla^4\bar{w} = 0$, the expansions truncate. A simple example of a simply supported plate under a sinusoidal load serves to illustrate the usefulness of the formulae that have been derived in this section. Suppose the plate occupies the region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and $-h/2 \leqslant z \leqslant h/2$ and is acted on by the transverse load

$$p = p_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \tag{43}$$

The boundary conditions for the simply supported edges are

$$w = 0, \quad w_{,xx} = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad a$$
$$w = 0, \quad w_{,yy} = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad b.$$

Kirchhoff's solution is

$$w^* = \frac{p_0}{\alpha^4 D} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{44}$$

where

$$\alpha = \pi\sqrt{(1/a^2 + 1/b^2)}.$$

The corresponding solution (35) is obtained by replacing ∇ by iα. Equation (35) thus yields

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(1 - \frac{\sinh(\alpha h)}{\alpha h}\right)^{-1} \frac{h^2\alpha^2}{6} \left\{ \frac{\sinh(\alpha z)}{\alpha} \left(\cosh(\alpha h/2) - \frac{\sinh(\alpha h/2)\alpha h}{4(1-\nu)}\right) \right.$$
$$\left. + \left(z \cosh(\alpha z) - \frac{\sinh(\alpha z)}{\alpha}\right) \frac{\cosh(\alpha h/2)}{2(1-\nu)} \right\} \frac{p_0}{\alpha^4 D} \begin{pmatrix} \pi/a \cos(\pi x/a) \sin(\pi y/b) \\ \pi/b \sin(\pi x/a) \cos(\pi y/b) \end{pmatrix}$$

and

$$w = -\left(1 - \frac{\sinh(\alpha h)}{\alpha h}\right)^{-1} \frac{\alpha^2 h^2}{6} \left\{ \cosh(\alpha z) \left(\cosh(\alpha h/2) + \frac{\sinh(\alpha h/2)\alpha h}{4(1-\nu)}\right) \right.$$
$$\left. - \frac{\alpha z \sinh(\alpha z)}{2(1-\nu)} \cosh(\alpha h/2) \right\} \frac{p_0}{\alpha^4 D} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \tag{45}$$

The vertical displacement at the middle of the top surface is thus predicted as

$$w_c = \frac{\alpha^3 h^3}{12} \frac{\cosh(\alpha h) + 1}{\sinh(\alpha h) - \alpha h} \frac{p_0}{\alpha^4 D}. \tag{46}$$

The vertical displacement in the middle of the midsurface is

$$w_0 = \frac{\alpha^3 h^3}{6} \left\{ \cosh(\alpha h/2) + \frac{\sinh(\alpha h/2)\alpha h}{4(1-\nu)} \right\} \frac{1}{\sinh(\alpha h) - \alpha h} \frac{p_0}{\alpha^4 D}$$

which for small h, for a square plate gives

$$w_0 = \left(1 + \frac{\pi^2}{20} \frac{(8-3\nu)}{(1-\nu)} \frac{h^2}{a^2} - \frac{\pi^4(227-157\nu)}{16\,800(1-\nu)} \frac{h^4}{a^4} + \cdots \right) \frac{p_0}{\alpha^4 D} \tag{47}$$

agreeing with Donnell (p. 244 of Ref. [9]). From the expressions for the moments in eqns (36) it can be seen that this solution has $M_x = M_y = 0$ on all the edges of the plate.

A uniform load p_0 can be represented by the Fourier expansion

$$p_0 = \frac{16p_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)(2n-1)} \sin \frac{(2m-1)\pi x}{a} \sin \frac{(2n-1)\pi y}{b} \tag{48}$$

so the displacement at the middle of the plate in this case is

$$w_0 = \frac{16p_0}{6\pi^2 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)(2n-1)} \alpha_{mn}^3 h^3 \left(\cosh (\alpha_{mn} h/2) + \frac{\sinh (\alpha_{mn} h/2) \alpha_{mn} h}{4(1-\nu)} \right) \frac{1}{\sinh (\alpha_{mn} h) - \alpha_{mn} h} \frac{1}{\alpha_{mn}^4} \tag{49}$$

and

$$\alpha_{mn} = \pi \sqrt{\left(\frac{(2m-1)^2}{a^2} + \frac{(2n-1)^2}{b^2} \right)}.$$

6. OTHER PLATE THEORIES

In most standard texts the precise assumptions that are made in deriving the equations for the various approximate theories of linear elastic plates are not made particularly clear. In many cases the stated approximations lead to contradictions. This section contains an attempt to clarify the situation and relates the various approximate theories to the exact results.

The Kirchhoff–Love theory can be termed an assumed strain theory as the governing equations are the strain displacement relations (1), the stress–strain relations

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu \\ -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix}, \quad \gamma_{xy} = \frac{1}{G} \tau_{xy} \tag{50}$$

which replace relations (2), the equilibrium equation, eqn (3), and the assumed strain condition

$$\begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \\ \varepsilon_z \end{pmatrix} = 0 \quad \text{or} \quad \begin{pmatrix} u_{,z} + w_{,x} \\ v_{,z} + w_{,y} \\ w_{,z} \end{pmatrix} = 0. \tag{51}$$

From eqns (51) and the symmetry conditions about $z = 0$ it can be deduced that w only depends on x and y and u and v are linear in z. Thus

$$\begin{aligned} w &= w(x, y) \\ u &= -z w_{,x} \\ v &= -z w_{,y}. \end{aligned} \tag{52}$$

These agree with the terms of lowest order in z, as given by eqn (12) if $(u', v', \bar{w}) = (-w_{,x}, -w_{,y}, w)$. It next follows from eqns (52), (50) and the strain displacement equations that

$$\begin{pmatrix} -z w_{,xx} \\ -z w_{,yy} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu \\ -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix}, \quad -2z w_{,xy} = \frac{2(1+\nu)}{E} \tau_{xy} \tag{53}$$

or

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{-Ez}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} w_{,xx} \\ w_{,yy} \\ w_{,xy} \end{pmatrix} \tag{54}$$

and so

$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \frac{-Eh^3}{12(1-\nu^2)} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} w_{,xx} \\ w_{,yy} \\ w_{,xy} \end{pmatrix}$$

from the equations for the moments.

Substitution of these expressions into eqn (15) yields finally the standard equation

$$D\nabla^4 w = p. \tag{55}$$

The remaining stress components τ_{xz}, τ_{yz} and σ_z can be determined from the equilibrium equation, eqn (3), once the solution for w is known. Thus

$$\begin{pmatrix} \tau_{xz} \\ \tau_{yz} \\ \sigma_z \end{pmatrix} = -\int_{-h/2}^{z} \begin{pmatrix} \sigma_{x,x}+\tau_{xy,y} \\ \tau_{xy,x}+\sigma_{y,y} \\ \tau_{xz,x}+\tau_{yz,y} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -p/2 \end{pmatrix} = \frac{E}{2(1-\nu^2)} \begin{pmatrix} (z^2-h^2/4)\nabla^2 w_{,x} \\ (z^2-h^2/4)\nabla^2 w_{,y} \\ -z(z^2/3-h^2/4)p/D \end{pmatrix}. \tag{56}$$

The corresponding shears are given by

$$\begin{pmatrix} q_x \\ q_y \end{pmatrix} = -D\nabla^2 \begin{pmatrix} w_{,x} \\ w_{,y} \end{pmatrix}. \tag{57}$$

A comparison with eqns (16) shows that in Kirchhoff's theory the leading terms in the shears are assumed to be zero, i.e. $u' = -\bar{w}_{,x}$ and $v' = -\bar{w}_{,y}$ so that $e = -2\nabla^2\bar{w}$ and eqn (57) are accurate to terms of $O(h^2)$.

In Mindlin's theory, which can also be regarded as an assumed strain theory, the governing equations are assumed to be the strain displacement equations, eqns (1), the stress–strain relations of Kirchhoff's theory, eqns (50), which replace the exact relations (2) as before, the equilibrium equation, eqn (3), and the strain conditions and strain–stress relations

$$\begin{pmatrix} \gamma_{xz,z} \\ \gamma_{yz,z} \\ \varepsilon_z \end{pmatrix} = 0$$

$$\begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \frac{1}{G'} \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix} \tag{58}$$

where

$$G' = K^2 G = \pi^2/12G.$$

In this theory it can again be deduced that w depends only on x and y and that u and v are linear in z. However, unknown functions appear in the expressions for u and v. Thus eqns (52) are replaced by

$$w = w(x,y)$$
$$u = -z\theta_x$$
$$v = -z\theta_y. \tag{59}$$

The stress–strain relationships are

$$-\begin{pmatrix} z\theta_{x,x} \\ z\theta_{y,y} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu \\ -\nu & 1 \end{pmatrix}\begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix}, \quad -z(\theta_{x,y}+\theta_{y,x}) = \frac{2(1+\nu)}{E}\tau_{xy} \tag{60}$$

or

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{-Ez}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}\begin{pmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \frac{1}{2}(\theta_{x,y}+\theta_{y,x}) \end{pmatrix} \tag{61}$$

and so

$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = -D\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}\begin{pmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \frac{1}{2}(\theta_{x,y}+\theta_{y,x}) \end{pmatrix}. \tag{62}$$

From the equations for the shears, eqns (58), and the strain displacement equations

$$\begin{pmatrix} q_x \\ q_y \end{pmatrix} = G'h\begin{pmatrix} w_{,x}-\theta_x \\ w_{,y}-\theta_y \end{pmatrix} = -D\begin{pmatrix} \theta_{x,xx}+\nu\theta_{y,xy}+\frac{1}{2}(1-\nu)(\theta_{x,yy}+\theta_{y,xy}) \\ \frac{1}{2}(1-\nu)(\theta_{x,xy}+\theta_{y,xx})+\nu\theta_{x,xy}+\theta_{y,yy} \end{pmatrix} \tag{63}$$

eqns (15) and (63) give

$$-p = G'h(\nabla^2 w-\theta_{x,x}-\theta_{y,y}) = -D\nabla^2(\theta_{x,x}+\theta_{y,y}).$$

The final equation for w is obtained by eliminating

$$-(\theta_{x,x}+\theta_{y,y}) = (M_x+M_y)/D(1+\nu)$$

from this last equation to find

$$G'h\left(\nabla^4 w-\frac{p}{D}\right) = -\nabla^2 p$$

or

$$\nabla^4 w = \frac{1}{D}\left(1-\frac{2h^2\nabla^2}{\pi^2(1-\nu)}\right)p. \tag{64}$$

In Mindlin's theory it is more usual to have the governing equations as three coupled second-order equations for w, θ_x and θ_y.

These are

$$(w_{,x}-\theta_x)_{,x}+(w_{,y}-\theta_y)_{,y}+\frac{24(1+\nu)}{\pi^2 Eh}p = 0 \tag{65}$$

and

$$w_{,x}-\theta_x+\frac{10}{\pi^2}\frac{h^2}{5(1-\nu)}\left[\theta_{x,xx}+\nu\theta_{y,xy}+\frac{(1-\nu)}{2}(\theta_{x,yy}+\theta_{y,yy})\right]=0$$

and

$$w_{,y}-\theta_y+\frac{10}{\pi^2}\frac{h^2}{5(1-\nu)}\left[\frac{(1-\nu)}{2}(\theta_{x,yx}+\theta_{y,xx})+\nu\theta_{x,xy}+\theta_{y,yy}\right]=0. \quad (66)$$

A comparison between eqns (63) and (16) shows that to leading order in h, G' should be equated to G. Mindlin's theory therefore effectively replaces

$$e_{,x} = -\nabla^2(\bar{w}_{,x}+u')+(u_{,y}+u_{,x})_{,y}+2u_{,xx}$$

by some multiple (involving a differential operator) of $\bar{w}_{,x}+u'$ and similarly $e_{,y} = -\nabla^2(\bar{w}_{,y}+v')+(u_{,y}+v_{,x})_{,x}+2v_{,yy}$ by the same multiple of $\bar{w}_{,y}+v'$. Mindlin's theory involves the modelling (neglect) of the in-plane strain gradients.

The previous two theories involve an assumed strain field. Reissner's theory on the other hand is an assumed stress theory. The basic governing equations are the strain displacement equations, eqns (1), the stress–strain relations

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\end{pmatrix}=\frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix},\quad \begin{pmatrix}\gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{pmatrix}=\frac{1}{G}\begin{pmatrix}\tau^*_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{pmatrix} \quad (67)$$

where τ^*_{xy} is a modified shear, the equilibrium equation, eqn (3), and the assumed stress field

$$\begin{pmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{pmatrix}=\begin{pmatrix}az\\ bz\\ cz\\ \frac{1}{2}d(-z^2+h^2/4)\\ \frac{1}{2}e(-z^2+h^2/4)\end{pmatrix} \quad (68)$$

where a–e are functions of x and y only. Functions a–e may be related to the bending moments and shears via the equations following eqns (50). Thus

$$\begin{pmatrix}M_x\\ M_y\\ M_{xy}\end{pmatrix}=\frac{h^3}{12}\begin{pmatrix}a\\ b\\ c\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}q_x\\ q_y\end{pmatrix}=\frac{h^3}{12}\begin{pmatrix}d\\ e\end{pmatrix}. \quad (69)$$

The remaining stress component satisfies

$$\frac{\partial\sigma_z}{\partial z}=-\frac{\partial\tau_{xz}}{\partial x}-\frac{\partial\tau_{yz}}{\partial y}=-\frac{6}{h^3}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right)\left(\frac{h^2}{4}-z^2\right)=\frac{6p}{h^3}\left(\frac{h^2}{4}-z^2\right)$$

so

$$\sigma_z=\frac{6p}{h^3}\left(\frac{h^2 z}{4}-\frac{z^3}{3}\right). \quad (70)$$

The strains may be deduced from eqns (67) as satisfying

$$\begin{pmatrix} u_{,x} \\ v_{,y} \end{pmatrix} = \frac{12}{h^3 E}\begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \end{pmatrix}\begin{pmatrix} zM_x \\ zM_y \\ zp\left(\dfrac{h^2}{8} - \dfrac{z^2}{6}\right) \end{pmatrix} \tag{71}$$

and

$$\begin{pmatrix} u_{,y} + v_{,x} \\ w_{,x} + u_{,z} \\ v_{,z} + w_{,y} \end{pmatrix} = \frac{2(1+\nu)12}{Eh^3}\begin{pmatrix} zM_{xy} + \dfrac{h^3}{12}(\tau^*_{xy} - \tau_{xy}) \\ \frac{1}{2}(h^2/4 - z^2)q_x \\ \frac{1}{2}(h^2/4 - z^2)q_y \end{pmatrix}. \tag{72}$$

As u and v are both cubic in z, $u_{,y}+v_{,x}$ will also in general be a cubic. The system will be inconsistent unless τ^*_{xy} differs from τ_{xy} by a term which is cubic in z. Reissner's theory involves mean values of the velocity fields defined by

$$\omega_x = -\frac{12}{h^3}\int_{-h/2}^{h/2} uz \, \mathrm{d}z, \quad \omega_y = -\frac{12}{h^3}\int_{-h/2}^{h/2} vz \, \mathrm{d}z$$

and

$$\bar{w} = \frac{12}{h^3}\int_{-h/2}^{h/2} \tfrac{1}{2} w\left(\frac{h^2}{4} - z^2\right) \mathrm{d}z. \tag{73}$$

From eqn (71)

$$\begin{pmatrix} \omega_{x,x} \\ \omega_{y,y} \end{pmatrix} = -\frac{12}{h^3 E}\begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \end{pmatrix}\begin{pmatrix} M_x \\ M_y \\ ph^2/10 \end{pmatrix}$$

or

$$\begin{pmatrix} M_x \\ M_y \end{pmatrix} = D\begin{pmatrix} 1 & \nu & 1+\nu \\ \nu & 1 & 1+\nu \end{pmatrix}\begin{pmatrix} -\omega_{x,x} \\ -\omega_{y,y} \\ 6p\nu/5Eh \end{pmatrix}. \tag{74}$$

Also

$$\omega_{x,y} + \omega_{y,x} = -\frac{12}{h^3}\int_{-h/2}^{h/2} (u_{,y} + v_{,x}) z \, \mathrm{d}z = \frac{-12}{h^3}\frac{2(1+\nu)12}{Eh^3}$$
$$\times \int_{-h/2}^{h/2} \left[z^2 M_{xy} + \frac{zh^3}{12}(\tau^*_{xy} - \tau_{xy}) \right] \mathrm{d}z = -\frac{12}{h^3}\frac{2(1+\nu)}{E} M_{xy} \tag{75}$$

provided

$$\int_{-h/2}^{h/2} z(\tau^*_{xy} - \tau_{xy}) \, \mathrm{d}z = 0$$

i.e. consistency requires that the moment of the actual shear equals the moment of the assumed shear. From eqns (74), (75) and the integrated equilibrium equations, eqns (15), it follows that

$$D\left(\omega_{x,xxx}+\omega_{x,xyy}+\omega_{y,yxx}+\omega_{y,yyy}-\frac{6\nu(1+\nu)}{5Eh}\nabla^2 p\right)-p=0. \tag{76}$$

Further relations connecting ω_x with the shears can be obtained. Thus

$$\frac{h^3}{12}\omega_x=-\int_{-h/2}^{h/2} uz\,\mathrm{d}z=-[u(z^2/2-h^2/8)]_{-h/2}^{h/2}+\int_{-h/2}^{h/2}\tfrac{1}{2}u_{,z}(z^2-h^2/4)\,\mathrm{d}z$$

$$=-\int_{-h/2}^{h/2}\tfrac{1}{2}(h^2/4-z^2)(\gamma_{xz}-w_{,x})\,\mathrm{d}z$$

$$=\frac{-2(1+\nu)/12}{Eh^3}\int_{-h/2}^{h/2}\left[\frac{\mathrm{d}z}{4}(h^2/4-z^2)^2\right]q_x+\bar{w}_{,x}$$

i.e.

$$\omega_x=-\frac{12}{5}\frac{1+\nu}{Eh}q_x+\bar{w}_{,x}. \tag{77}$$

There is a similar expression for ω_y. The expressions for the shears are thus

$$\begin{pmatrix}q_x\\ q_y\end{pmatrix}=\frac{5Eh}{12(1+\nu)}\begin{pmatrix}\bar{w}_{,x}-\omega_x\\ \bar{w}_{,y}-\omega_y\end{pmatrix}. \tag{78}$$

Comparing eqns (63), (77) and (78) shows the differences between Mindlin and Reissner's theories, namely that

$$G'h=\frac{\pi^2}{24}\frac{Eh}{1+\nu}$$

is replaced by $(5/12)(Eh/(1+\nu))$, w is replaced by $\bar{w}$ and (θ_x,θ_y) by (ω_x,ω_y).

Substitution of eqns (77) and (78) into eqn (76) yields after some manipulation

$$D\nabla^4\bar{w}=\left(1-\frac{2-\nu}{1-\nu}\frac{h^2}{10}\nabla^2\right)p. \tag{79}$$

The three second-order equations for $\bar{w}$, ω_x and ω_y are obtained from the integrated form of the equilibrium equations, eqns (31) and (78) as

$$(\bar{w}_{,x}-\omega_x)_{,x}+(\bar{w}_{,y}-\omega_y)_{,y}+\frac{24(1+\nu)}{10Eh}p=0 \tag{80}$$

$$(\bar{w}_{,x}-\omega_x)+\frac{h^2}{5(1-\nu)}\left(\omega_{x,xx}+\frac{1+\nu}{2}\omega_{y,xy}+\frac{1-\nu}{2}\omega_{x,yy}\right)-\frac{6\nu h(1+\nu)}{25E(1-\nu)}p_{,x}=0$$

and

$$(\bar{w}_{,y}-\omega_y)+\frac{h^2}{5(1-\nu)}\left(\omega_{y,yy}+\frac{1+\nu}{2}\omega_{x,xy}+\frac{1-\nu}{2}\omega_{y,xx}\right)-\frac{6\nu h(1+\nu)}{25E(1-\nu)}p_{,y}=0. \tag{81}$$

A comparison of eqns (79) and (64) shows that a correction factor of $(2-\nu)/10$ has replaced Mindlin's value of $2/\pi^2=2.03$. In a further theory due to Hencky (cf. p. 47 of Ref. [13]) a factor of 1/6 appears. From eqns (18) the right-hand side of Cheng's equation for small h^2 is

$$\left(1-\frac{(8-3\nu)h^2\nabla^2}{40(1-\nu)}\right)p$$

so the "correct" correction factor is $(1/10)(2-3/4\nu)$.

According to Kromm (cf. Chap. 5 of Ref. [13]) the factor is $F = 2(1-E(\alpha))/a^2E(\alpha)$ where $E(\alpha) = (24/\alpha^3)(\alpha/2 - \tan h(\alpha/2))$ and α has a non-negative value. F is bounded by 1/6 and 1/5, whereas Cheng's value is bounded by $(13/80) = (1/6)(39/40)$ and 1/5 for $0 \leqslant \nu \leqslant \frac{1}{2}$.

7. RELATION BETWEEN MINDLIN AND KIRCHHOFF'S VARIABLES

It seems not to have been observed that there is a strong connection between Kirchhoff and Mindlin's formulations. Let w_k be a solution of eqn (19) for some set of boundary conditions, i.e.

$$\nabla^4 w_k = \frac{p}{D} \tag{82}$$

then

$$w_m = \left(1-\frac{2h^2\nabla^2}{\pi^2(1-\nu)}\right)w_k + c \tag{83}$$

satisfies the corresponding Mindlin equation, eqn (64), provided $\nabla^4 c = 0$. From eqn (65)

$$(w_{k,x}-\theta_x)_{,x} + (w_{k,y}-\theta_y)_{,y} + \nabla^2 c = 0$$

which is satisfied if

$$\begin{pmatrix}\theta_x\\ \theta_y\end{pmatrix} = \begin{pmatrix}w_{k,x}\\ w_{k,y}\end{pmatrix} \tag{84}$$

and $\nabla^2 c = 0$.

Equations (83) and (84) then reduce eqn (66) to

$$c_{,x} = c_{,y} = 0.$$

So Mindlin and Kirchhoff's solutions are connected by eqns (83) and (84) provided c is a constant for given, E, ν and h dependent on the boundary conditions. There does not appear to be such a simple relationship connecting Kirchhoff and Reissner's equations.

The simplest examples of this result are beam problems. We give below one such example, that of a cantilever with uniform loading.

Kirchhoff's solution w_k satisfies

$$\frac{d^4 w_k}{dx^4} = \frac{q}{EI}, \quad 0 \leqslant x \leqslant l \tag{85}$$

subject to

$$w_k(0) = \frac{dw_k}{dx}(0) = 0$$

and

$$\frac{d^3 w_k}{dx^3}(l) = \frac{d^2 w_k}{dx^2}(l) = 0.$$

These equations have the solution

$$w_k = \frac{q}{EI}\frac{1}{24}(x^4 - 4lx^3 + 6l^2x^2)$$

$$\frac{dw_k}{dx} = \frac{q}{EI}\frac{1}{24}(4x^3 - 12lx^2 + 12l^2x).$$

Mindlin's solutions w_m, θ satisfy

$$\frac{GA}{\alpha_s}\frac{d}{dx}\left(\frac{dw_M}{dx} - \theta\right) = -q, \quad 0 \leqslant x \leqslant l$$

and

$$\frac{GA}{\alpha_s}\left(\frac{dw_M}{dx} - \theta\right) = -EI\frac{d^2\theta}{dx^2}, \quad 0 \leqslant x \leqslant l \tag{86}$$

where A is the cross-sectional area and α_s is the shear factor, subject to

$$w_m(0) = \theta(0) = 0$$

and

$$\left(\frac{dw_M}{dx} - \theta\right)(l) = \frac{d\theta}{dx}(l) = 0.$$

These have the solutions from eqns (83) and (84), or otherwise

$$w_M = \frac{q}{EI}\frac{1}{24}(x^4 - 6lx^3 + 4l^2x^2) - \frac{q\alpha_s}{2GA}(x^2 - 2lx)$$

$$\theta = \frac{q}{EI}\frac{1}{24}(4x^3 - 12lx^2 + 8l^2x).$$

In the beam cases it can be seen how the relationship is derived by integrating eqns (87). It is also evident that for simple loading conditions the formulae are easily applied.

For plate problems few analytic solutions are available on which the formulae may be used. One example is that of a circular plate, clamped on its edge with an applied uniform loading.

Kirchhoff's solution w_k satisfies

$$\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right)^2 w_k = \frac{p}{D} \tag{87}$$

subject to

$$w_k(a) = \frac{\partial w_k}{\partial r}(a) = \frac{\partial w_k}{\partial r}(0) = 0$$

and $w_k(0)$ finite.

The solution is

$$w_k = \frac{p}{64D}(a^2 - r^2)^2$$

and

$$\frac{\partial w_k}{\partial r} = -\frac{p}{64D}4r(a^2 - r^2).$$

Mindlin's solutions w_M, θ_r satisfy the polar forms of eqns (65), (66) subject to

$$w_M(a) = \theta(a) = \frac{\partial w_M}{\partial r}(0) = 0$$

$$w_M(0) \text{ finite.}$$

These have solutions, again from eqns (83) and (84), given by

$$w_M = \frac{p}{64D}(a^2 - r^2)^2 + \frac{p}{64F}16(a^2 - r^2)$$

$$\theta = \frac{-p}{64D}4r(a^2 - r^2)$$

where

$$F = \frac{\pi^2 Eh}{24(1+\nu)}.$$

Allowing for the different shear factor this is identical to the result obtained by Reissner and stated in Timoshenko and Woinowsky-Kreiger[12].

8. CONCLUSIONS

In this paper it has been shown how the exact solutions of the full linear elastic equations can be written in terms of a Kirchhoff boundary value problem. Many of the formulae derived are also applicable to a plate of non-uniform thickness provided the elastic material is confined to lie between two antisymmetrically loaded, shear free surfaces $z = \pm f(x, y)$. It has also been shown that the plate theories of Mindlin and Reissner are approximations modelling the in-plane strain gradients. Because the exact theory shows that the precise way in which the general solution depends on the plate thickness via terms involving h^3 and h^2 for any given values of E and ν it is also possible to obtain Kirchhoff solutions from Mindlin or Reissner's approximations. As the precise connection between the various approximate solutions and the exact solution is clear and stable numerical methods based on finite elements are available for Mindlin plates these numerical solutions may be utilized with more confidence.

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